SECTION 6.1: NET AND TOTAL CHANGE

RECALL: If f is continuous on $[a, b]$ and F is any antiderivative of f, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

If f' is continuous, then f is an antiderivative of f' (why?) so we can write this as:

$$\int_a^b f'(x)\,dx = f(b) - f(a) = \text{net change in } f \text{ from } x = a \text{ to } x = b,$$

which means *integrating a rate function gives a net change.*

EXAMPLE 1: The area enclosed by a circle of radius r is $A(r) = \pi r^2$. Write and evaluate an integral which determines the change in area enclosed by a circle as the radius is changed from $r = 2$ inches to $r = 5$ inches.

Since $A'(r) = 2\pi r$ (does this formula look familiar?), we have $\int_2^5 2\pi r\,dr = \pi r^2\Big|_{r=2}^{r=5} = \pi(5)^2 - \pi(2)^2 = 21\pi$ inches^2

It is often helpful to rewrite $\int_a^b f'(x)\,dx = f(b) - f(a)$ as $\int_a^x f'(u)\,du = f(x) - f(a)$ and solve for $f(x)$:

$$f(x) = f(a) + \int_a^x f'(u)\,du$$

or, said differently:

$$f(x) = \text{value of } f \text{ at } a + \text{change in } f \text{ from } a \text{ to } x$$

EXAMPLE 2: The population growth rate of Sasquatch (in Sasquatch per year) is $P'(t) = 5 - t$, $t \geq 0$.

1. What will be the change in Sasquatch population over the first 16 years?

$$\int_0^{16} P'(t)\,dt = \int_0^{16} (5 - t)\,dt = 5t - \frac{1}{2}t^2\Big|_{t=0}^{t=16} = -48$$

   Hence, the population will decrease by 48 Sasquatch over the first 16 years.

2. If there are 250 Sasquatch present initially, how many will remain in 16 years?

   In 16 years there will be $250 + (-48) = 202$ Sasquatch.

3. Find a formula for population of Sasquatch, $P(t)$.

$$\begin{aligned}
P(t) &= P(0) + \int_0^t P'(u)\,du \\
&= 250 + \int_0^t (5 - u)\,du \\
&= 250 + \left(5u - \frac{1}{2}u^2\right)\Big|_{u=0}^{u=t} \\
&= 250 + 5t - \frac{1}{2}t^2 \\
P(t) &= -\frac{1}{2}t^2 + 5t + 250
\end{aligned}$$

DISPLACEMENT AND TOTAL DISTANCE TRAVELED

RECALL: Given a position function for an object, $s(t)$:

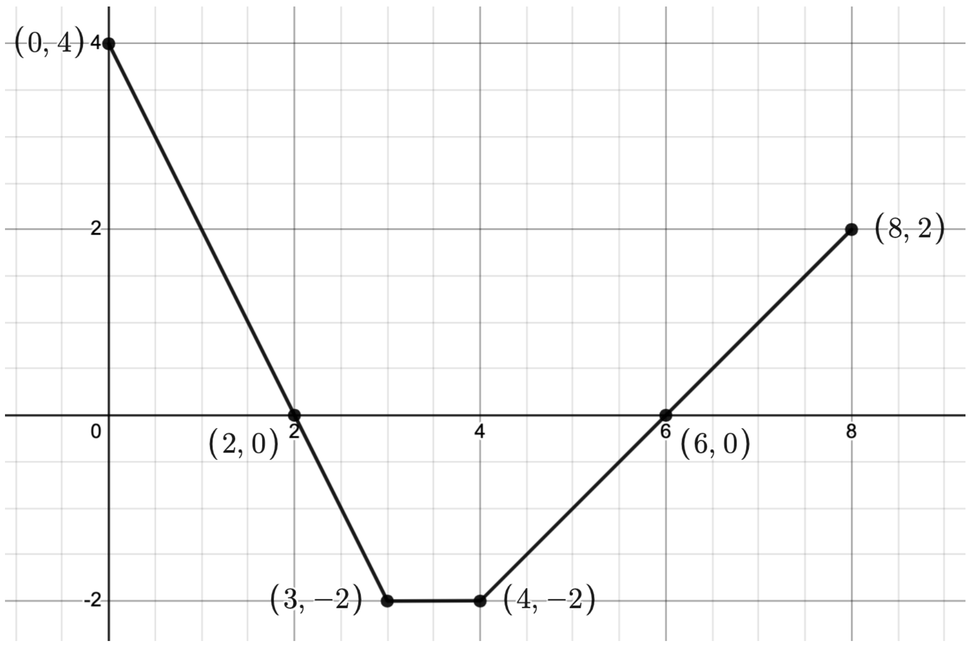
- the velocity of the object is $v(t) = s'(t)$
- the speed of the object is $|v(t)| = |s'(t)|$

Hence:

$$\int_a^b v(t)\,dt = \int_a^b s'(t)\,dt = s(b) - s(a) = \text{the displacement over the interval [a,b]}$$

$$\int_a^b |v(t)|\,dt = \int_a^b |s'(t)|\,dt = \text{the total distance traveled over the interval [a,b]}$$

EXAMPLE 3: An object moves along the y-axis with velocity $v(t)$, graphed below, measured in feet per second.



$y = v(t)$

1. Find and interpret $\int_0^4 v(t)\,dt$

2. Find and interpret $\int_0^4 |v(t)|\,dt$

3. Find and interpret $\int_4^8 v(t)\,dt$

4. Find and interpret $\int_4^8 |v(t)|\,dt$

5. If the object is initially at $(0, 1)$, where is it when $t = 4$? $t = 8$?

6. Find the total distance travelled over the interval $[0, 8]$.

EXAMPLE 4: An object travels up and down the y-axis with velocity $v(t) = 4\sin(3t)$, $t \geq 0$.

1. Find and the displacement over the interval $\left[0, \frac{\pi}{2}\right]$.

$$\int_0^{\frac{\pi}{2}} 4\sin(3t)\, dt = -\frac{4}{3}\cos(3t)\Big|_{t=0}^{t=\frac{\pi}{2}} = \left(-\frac{4}{3}\cos\left(\frac{3\pi}{2}\right)\right) - \left(-\frac{4}{3}\cos(0)\right) = \frac{4}{3}$$

This means when $t = \frac{\pi}{2}$, the object is $\frac{4}{3}$ units above **above** where it was at $t = 0$.

2. Find the total distance traveled over $\left[0, \frac{\pi}{2}\right]$.

Total distance traveled is: $\displaystyle\int_0^{\frac{\pi}{2}} |4\sin(3t)|\, dt = 4\int_0^{\frac{\pi}{2}} |\sin(3t)|\, dt$.

We need to make a sign diagram for $f(t) = \sin(3t)$ over $\left[0, \frac{\pi}{2}\right]$.

	$(+)$		$(-)$		
		0			$f(t)$
0		$\frac{\pi}{3}$		$\frac{\pi}{2}$	t

Hence, on $\left[0, \frac{\pi}{3}\right]$, $\sin(3t) \geq 0$ so $|\sin(3t)| = \sin(3t)$ and on $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$, $\sin(3t) < 0$ so $|\sin(3t)| = -\sin(3t)$.

Using the additive interval property, we get:

$$\begin{aligned} 4\int_0^{\frac{\pi}{2}} |\sin(3t)|\, dt &= 4\int_0^{\frac{\pi}{3}} |\sin(3t)|\, dt + 4\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} |\sin(3t)|\, dt \\ &= 4\int_0^{\frac{\pi}{3}} \sin(3t)\, dt - 4\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(3t)\, dt = \ldots = 4 \end{aligned}$$

The total distance traveled is 4 units.

3. If the object is initially at $(0, 5)$, where is it at $t = \frac{\pi}{2}$?

We know the displacement of the object over the interval $\left[0, \frac{\pi}{2}\right]$ is $\frac{4}{3}$.

So when $t = \frac{\pi}{2}$, the object is $\frac{4}{3}$ units above where it was when $t = 0$ at $\left(0, 5 + \frac{4}{3}\right) = \left(0, \frac{19}{3}\right)$.

IN GENERAL:

$$\int_a^b f'(x)\,dx = f(b) - f(a) = \text{net change in } f \text{ from } x = a \text{ to } x = b,$$

$$\int_a^b |f'(x)|\,dx = \text{total change in } f \text{ from } x = a \text{ to } x = b,$$

EXAMPLE 5: The profit $P(x)$, in dollars, obtained by selling x game consoles satisfies: $P'(x) = -2x + 170$.

1. Find and interpret: $\displaystyle\int_0^{100} P'(x)\,dx$.

$$\int_0^{100} P'(x)\,dx = \int_0^{100} (-2x+170)\,dx = -x^2 + 170x\Big|_{x=0}^{x=100} = \ldots = 7000$$

This means there is a \$7000 increase in profit over the course of selling 100 game consoles.

2. If the fixed costs in this scenario are \$5000, what is the profit obtained when selling 100 consoles?

The fixed costs are \$5000 means $P(0) = -5000$. Hence, $P(100) = P(0) + 7000 = -5000 + 7000 = 2000$.

The profit obtained by selling 100 consoles is \$2000.

3. Find and interpret: $\displaystyle\int_0^{100} |P'(x)|\,dx$.

Making a Sign Diagram for $P'(x) = -2x + 170$ over the interval $[0, 100]$ gives:

$$\begin{array}{ccccc} & (+) & 0 & (-) & P'(x) \\ 0 & & 85 & & 100 \quad x \end{array}$$

Since $P'(x) \ge 0$ on $[0, 85]$, $|P'(x)| = P'(x) = -2x + 170$.

Likewise, $P'(x) \le 0$ on $[85, 100]$, $|P'(x)| = -P'(x) = -(-2x + 170) = 2x - 170$.

Hence,

$$\begin{aligned} \int_0^{100} |P'(x)|\,dx &= \int_0^{85} |P'(x)|\,dx + \int_{85}^{100} |P'(x)|\,dx \\ &= \int_0^{85} (-2x+170)\,dx + \int_{85}^{100} (2x-170)\,dx \\ &= 7225 + 225 = 7450 \end{aligned}$$

This means the total change in profit over the course of selling 100 consoles is \$7450.

NOTE: Looking at the Sign Diagram for $P'(x)$ above, we see that profit is maximized selling 85 consoles.

Indeed we see that the change in profit over the course of selling the first 85 consoles is \$7225.

We then lose \$225 in profit when selling between 85 and 100 consoles for a net profit of \$7000.

HOMEWORK: Section 6.1: 7, 13 - 53 odd, 62*, 63*